A Curiously Rational Result

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The following is problem 1196 from the Spring 2009 Pi Mu Epsilon Journal [1]: Let $\mathbb{Q}^* = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, a \neq 0, b > 0, \text{ and GCD}\,(a,b) = 1\}$. In other words, $\mathbb{Q}^*$ is the set of all nonzero rational numbers written in lowest terms. Find, with proof, the value of

$$\sum_{\frac{a}{b} \in \mathbb{Q}^*} \frac{1}{(ab)^2}.$$

Proof. In order to find a general solution to this problem we will define a new function

$$g(m) = \sum_{\frac{a}{b} \in \mathbb{Q}^+} \frac{1}{(ab)^m},$$

where m is an even positive integer, and where $\mathbb{Q}^+$ is defined by removing the negative values of a in $\mathbb{Q}^*$. In other words, $\mathbb{Q}^+ = \{\frac{a}{b} \mid a, b \in \mathbb{N} \text{ and GCD}\,(a,b) = 1\}$. Then, replacing the 2 in our original problem with the general m,

$$\sum_{\frac{a}{b} \in \mathbb{Q}^*} \frac{1}{(ab)^m} = 2 \sum_{\frac{a}{b} \in \mathbb{Q}^+} \frac{1}{(ab)^m} = 2g(m).$$

Now, let

$$ab = n = \prod_{i=1}^{r} p_i^{\alpha_i},$$

where $r = \omega(n)$ is equal to the number of distinct primes in the factorization of n. We must find the number of choices of a and b such that $ab = n$ and GCD $(a,b) = 1$. Since $\prod p_i^{\alpha_i} = n = ab$, $\;p_i^{\alpha_i}$ can be in the factorization of either a or b. So, since there are r primes, the multiplication principle implies that there are $2^{\omega(n)}$ choices for a and b. Using these we conclude that

$$g(m) = \sum_{\frac{a}{b} \in \mathbb{Q}^+} \frac{1}{(ab)^m} = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^m}.$$

Now, we will derive an identity of the Riemann zeta function. First, we need few tools to work with. The following are taken from [2].

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Definition 1. *The Riemann zeta-function, $\zeta(s)$, is defined as*

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad \mathrm{Re}(s) > 1.$$

Definition 2. *An arithmetic function, $f(n)$, is multiplicative if, given $a, b \in \mathbb{N}$ and* $\mathrm{GCD}(a,b) = 1$, *$f(a)f(b) = f(ab)$.*

Identity 1. *It was observed by Euler that*

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p} \left(1 - p^{-s}\right)^{-1},$$

when $\mathrm{Re}(s) > 1$.

Theorem 1. *If $f(n)$ is a real or complex-valued multiplicative function such that $\sum_{n=1}^{\infty} |f(n)| < \infty$, then*

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \left(1 + f(p) + f\left(p^2\right) + f\left(p^3\right) + \ldots\right),$$

where $\prod_p$ is the product over all of the primes.

Using these, we can begin the derivation of our identity. Let $f(n) = \frac{2^{\omega(n)}}{n^s}$ where $\mathrm{Re}(s) > 1$. Suppose there exists $k, l \in \mathbb{N}$ such that $\mathrm{GCD}(k,l) = 1$. Then since k and l share no primes, kl contains all of the primes from k and l in its factorization. So $\omega(k) + \omega(l) = \omega(kl)$. Then

$$f(k)\,f(l) = \frac{2^{\omega(k)}}{k^s}\,\frac{2^{\omega(l)}}{l^s} = \frac{2^{\omega(k)+\omega(l)}}{k^s l^s} = \frac{2^{\omega(kl)}}{(kl)^s} = f(kl).$$

So, $f(n)$ is a multiplicative function.

Now, we observe that the following inequalities hold for all $p \geq 5$ and all $k \geq 1$: $2 \leq 2\sqrt{2^k}$, $2 \leq 2\sqrt{3^k}$, $2 \leq \sqrt{5^k}$, $2 \leq \sqrt{7^k}, \ldots,$ $2 \leq \sqrt{p^k}$. So, for $n = \prod p_i^{\alpha_i}$, by multiplying the previous inequalities together for the primes in n, we can see that $2^{\omega(n)} \leq 4\sqrt{n}$. So

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} < \sum_{n=1}^{\infty} \frac{4\sqrt{n}}{n^s} = \sum_{n=1}^{\infty} \frac{4}{n^{s-\frac{1}{2}}}.$$

Thus, since $\sum_{n=1}^{\infty} \frac{4}{n^{s-\frac{1}{2}}}$ converges for $\mathrm{Re}(s) > \frac{3}{2}$, $\sum_{n=1}^{\infty} |f(n)| < \infty$. It can be shown that $\sum f(n)$ converges for $\mathrm{Re}(s) > 1$. However, this is unnecessary for our proof. Now we can use Theorem 1.

This leads us to the following algebraic manipulations and substitutions:

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} &= \prod_p \left(1 + \frac{2^{\omega(p)}}{p^s} + \frac{2^{\omega\left(p^2\right)}}{p^{2s}} + \frac{2^{\omega\left(p^3\right)}}{p^{3s}} + \dots\right) \\
&= \prod_p \left(1 + \frac{2}{p^s}\left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots\right)\right) \\
&= \prod_p \left(1 + \frac{2}{p^s}\left(\frac{1}{1 - p^{-s}}\right)\right) \\
&= \prod_p \left(1 - p^{-s}\right)^{-1} \left(1 + p^{-s}\right) \\
&= \zeta\left(s\right) \prod_p \frac{1 - p^{-2s}}{1 - p^{-s}} \\
&= \frac{\zeta^2\left(s\right)}{\zeta\left(2s\right)}.
\end{aligned}
$$

So

$$
\sum_{n=1}^{\infty} f\left(n\right) = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2\left(s\right)}{\zeta\left(2s\right)}.
$$

Now returning to our original problem, recall that the summation can be expressed in general as $2g(m)$. So

$$
2g(m) = 2 \sum_{\frac{a}{b} \in \mathbb{Q}^+} \frac{1}{\left(ab\right)^m} = 2 \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^m} = 2 \frac{\zeta^2\left(m\right)}{\zeta\left(2m\right)}.
$$

The formula for calculating the Riemann zeta-function for even, positive integers can be found in [2].

Definition 3. *The Bernoulli numbers, B_j, are defined by*

$$
\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!} \quad \left(|z| < 2\pi\right).
$$

Theorem 2. *If $k \in \mathbb{N}$, and B_j denotes the jth Bernoulli number then,*

$$
\zeta\left(2k\right) = \frac{\left(-1\right)^{k+1} \left(2\pi\right)^{2k} B_{2k}}{2\left(2k\right)!}.
$$

Given these, we can calculate the solutions for positive even values. Let $m = 2k$, then

$$
2 \frac{\zeta^2\left(2k\right)}{\zeta\left(4k\right)} = -\left(\frac{\left(4k\right)!}{\left(2k\right)!^2}\right) \left(\frac{\left(B_{2k}\right)^2}{B_{4k}}\right).
$$

At first glance it may seem like this should always result in a negative value. However, after consideration, it can be seen that every Bernoulli number of the form B_{4k} is negative, so the above expression has a positive value. Taking $k = 1$ we obtain a solution to our original problem:

$$2g(2) = \sum_{\frac{a}{b} \in \mathbb{Q}^*} \frac{1}{(ab)^2} = 2\frac{\zeta^2(2)}{\zeta(4)} = -\left(\frac{(4)!}{(2)!^2}\right)\left(\frac{(B_2)^2}{B_4}\right) = -\left(\frac{24}{4}\right)\left(-\frac{30}{36}\right) = 5.$$

For $k = 2$, $k = 3$, and $k = 4$ respectively,

$$\sum_{\frac{a}{b} \in \mathbb{Q}^*} \frac{1}{(ab)^4} = \frac{7}{3} \quad \sum_{\frac{a}{b} \in \mathbb{Q}^*} \frac{1}{(ab)^6} = \frac{1430}{691} \quad \sum_{\frac{a}{b} \in \mathbb{Q}^*} \frac{1}{(ab)^8} = \frac{7293}{3617}.$$

□

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References

[1] "Problem 1196." *Pi Mu Epsilon Journal*, (559). Spring 2009. <`http://www.pme-math.org/journal/ProblemsS2009.pdf`>. 7 July 2009.

[2] Ivić, Aleksandar, *The Riemann Zeta-Function,* John Wiley & Sons, Inc., 1985.